Invited paper

Methods of Step-Size Distribution Optimization Used in S-SSFM Simulations of WDM Systems

Marek Jaworski

Abstract—Brief review of methods used for simulation of signal propagation in wavelength division multiplexed (WDM) links is presented. We propose two novel methods of step-size distribution optimization used to improve symmetrized split step Fourier method (S-SSFM) numerical efficiency: pre-simulated local error S-SSFM (PsLE S-SSFM) and modified logarithmic (ML S-SSFM). The PsLE S-SSFM contains two stages: in the initial stage step-size distribution optimization is carried out by combining local error method and pre-simulation with signal spectrum averaging; in the second stage conventional SSFM is used by applying optimal step-size distribution obtained in the initial stage. The ML S-SSFM is generalization of logarithmic method proposed to suppress spurious FWM tones, in which a slope of logarithmic step-size distribution is optimized. Overall time savings exceed 50%, depending of a simulated system scenario.

Keywords— local error method, logarithmic step, simulation, split step Fourier method, WDM systems.

1. Introduction

Split step Fourier method (SSFM) is commonly used for simulating of light propagation in an optical fibre, described by the nonlinear Schrödinger equation (NLSE) [1], due to its high numerical efficiency. Optimization of simulation time and accuracy is considered in many publications [2]–[12]. Higher order numerical methods (i.e., explicit Adams-Bashforth and implicit Adams-Moulton, etc.) or predictor-corrector methods [2] are used when the highest accuracy is needed. In this case the numerical effectiveness is better than for conventional symmetrized SSFM (S-SSFM). These methods are especially useful for simulations of soliton propagation, where linear (L) and nonlinear (N) operators in SSFM are self-balanced.

Typically, there are higher dispersion and lower nonlinearity in wavelength division multiplexed (WDM) transmission, when comparing to soliton transmission. As a consequence, special tailored methods should be applied for simulation of signal propagation in WDM links. In this case, S-SSFM is especially effective. It is a method of order $O(h^2)$, which is adequate for relatively low accuracy required (of the order of $10^{-2} - 10^{-3}$). Besides common used S-SSFM, another methods are used in special cases, e.g., split step wavelet collocation is faster then S-SSFM in very wideband simulations [3], but is applicable only for zero dispersion slope ($\beta_3 = 0$).

Modern WDM systems contain large number of channels and occupy very wide bandwidth, which cause difficulties in simulations due to spurious four wave mixing (FWM) and walk-off effect. Two class of methods are distinguished: single-band [1]–[7], [12] – in which full-bandwidth of WDM transmission is simulated, and multi-band [8]–[11] – in which separate channels are simulated by taking into consideration an influence of adjacent channels (Fig. 1). The single-band methods give an exact solution of the nonlinear Schrödinger equation, including the impact of nonlinear phenomena, like: self-phase modulation (SPM), cross-phase modulation (XPM) and FWM, but on the other hand, these methods are used mainly in narrow bandwidth cases due to its high simulation time. The multi-band methods are faster and more flexible, but give only limited information of nonlinear phenomena (i.e., SPM, XPM but not FWM) derived from other channels.

An optimal step-size in S-SSFM is of uttermost importance to improve the numerical efficiency. Local error method (LEM) is especially useful for step-size optimization, because it automatically adjusts simulation step for required accuracy [5]. In this method step-size is selected by calculating the relative local error δ_L of each single step, taking into account the error estimation and linear extrapolation. LEM provides higher accuracy than S-SSFM method, because it is of order $O(h^3)$. Simulations are conducted simultaneously with coarse ($2h$) and fine (h) steps, which needs additional 50% operations comparing with S-SSFM. Different multi-band methods have been evaluated in [11] and application of LEM method to XPM simulation in place of fixed-step was proposed, which improves simulation accuracy and efficiency up to 30%.

Lately, methods known in quantum mechanics was used for step-size calculation [4]. The optimal step-size $h_{optimal}$ can be estimated analytically for required global error δ_G. This procedure is fast in the case of lossless fiber. In a more realistic case with lossy fiber, the optimal step-size can be estimated as well, but with an additional computational effort [4].

In pre-simulation method the step-size is selected by calculating the global error δ_G in a series of fixed-step S-SSFM pre-simulations with signal spectrum averaging [6]:

$$\left|U_n^{red}\right| = \sqrt{\sum_{i=n\cdot N_{red}}^{n\cdot N_{red}+N_{red}-1} \left|U_i\right|^2},$$

$$\arg(U_n^{red}) = \arg\left[\sum_{i=n\cdot N_{red}}^{n\cdot N_{red}+N_{red}-1} \left(\left|U_i\right| \cdot U_i\right)\right], \qquad (1)$$

where: $U = \Im(u)$ is the Fourier transform of the original discrete signal with N samples, N_{red} is the reduction ratio,

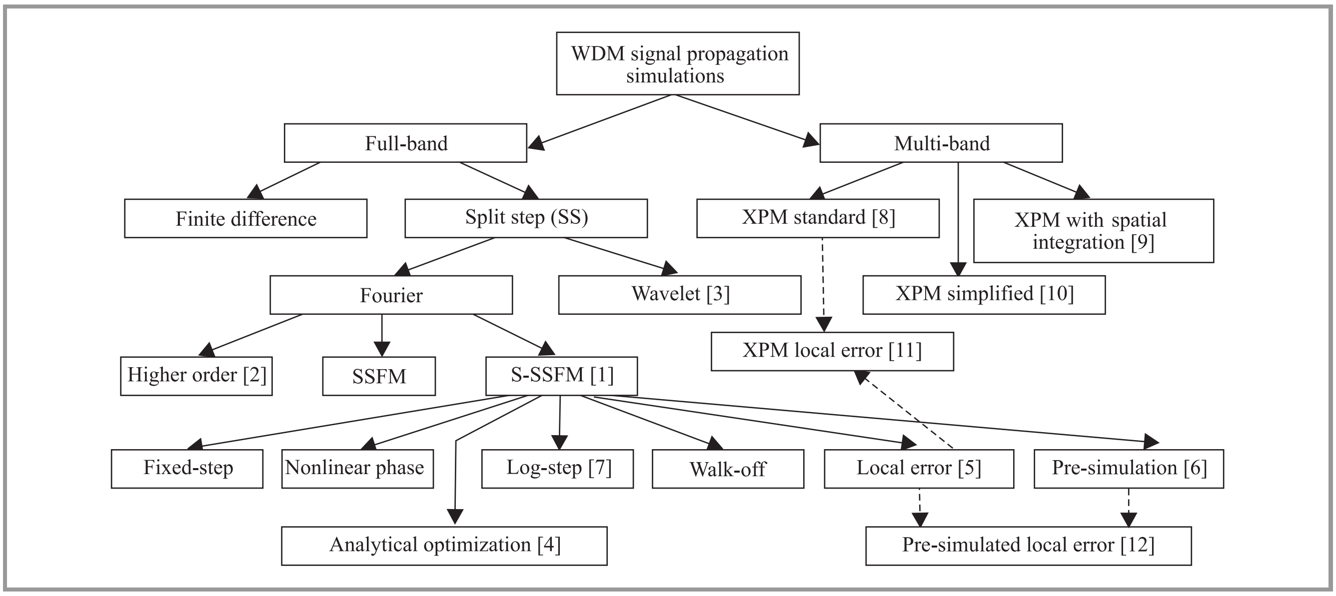

Fig. 1. Review of WDM signal propagation simulation methods.

and $n = -N/(2N_{red}),\ -N/(2N_{red})+1,\ldots,N/(2N_{red})-1$. For a given N_{red}, split step pre-simulation of the test signal can be much faster ($> N_{red}$) than the corresponding simulation of the original signal. Several pre-simulations must be carried-out iteratively to calculate optimal step-size $h_{optimal}$, required to achieve desired global accuracy. Pre-simulations typically takes 30% of full spectrum simulation time [6].

2. Pre-simulated Local Error S-SSFM

We proposed novel simulation method which comprises two stages: step-size optimization is carried out in the initial stage, combining local error and pre-simulation methods and in the second stage conventional S-SSFM is used by applying optimal step-size distribution $h_{optimal}(z)$, obtained in the initial stage. Overall time savings up to 50% are realistic, depending of simulated system scenario. We called this novel procedure pre-simulated local error S-SSFM (PsLE S-SSFM).

Modified LEM algorithm with averaged signal spectrum Eq. (1) is used in PsLE S-SSFM. Method of order $O(h^3)$ is utilized in [5] by combining a fractions of coarse u_c and fine u_f solutions to calculate the next step. In our method only fine solution u_f is used in pre-simulation and u_c is utilized only to calculate local error, which gives better stability and does not degrade accuracy considerably in the case of WDM simulations, where the global error δ_G is low – of the order of 10^{-3}. Contrary to original pre-simulation method [6], the duration of the initial stage is only a small percentage (2%) of the second stage, in which the full-band simulation is carried out.

Results. We have explored the applicability of PsLE S-SSFM method to WDM systems with different number of channels. The method was used for simulation of WDM link with the following parameters: RZ modulation format, bit rate of 40 Gbit/s, channel spacing of 100 GHz, channel power of 1 mW, simulated bandwidth of 320 GHz/channel and bit sequence length of 2^9, and various number of channels. Transmission line comprises two types of fiber, with parameters given in Table 1.

Table 1
Fiber parameters used in the simulations

Parameter [unit]	SSMF1	SSMF2
Length [km]	100	100
Attenuation [dB/km]	0.22	0.22
Dispersion [ps/(nm·km)]	16.00	5.00
Dispersion slope [ps/(nm·km)^2]	0.08	0.00
Nonlinear coefficient [1/(W·km)]	1.32	1.32

Results shown in Fig. 2 indicate that the PsLE S-SSFM is up to 50% faster than the walk-off method in all simulated

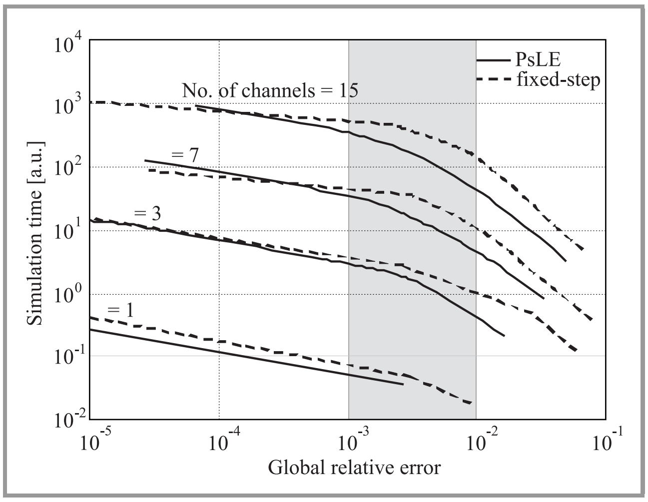

Fig. 2. Simulation time versus global relative error for fixed-step (dashed line) and PsLE (solid line) methods.

cases, in critical global error range of $10^{-2}-10^{-3}$. Relation between the method parameter and the global error was considered for fixed-step and PsLE methods (Fig. 3). The parameter of method is a parameter in a split step method that should be varied to obtain required accuracy. For required global error $\delta_G = 10^{-3}$ the local error (i.e., the parameter of PsLE method) varies from $2 \cdot 10^{-5}$ to $3 \cdot 10^{-4}$ for different number of simulated channels, in the same conditions the step-size (i.e., the parameter of fixed-step method) varies in a much wider range – from 8 m to 5000 m. As a rule of thumb, the global relative error equals $\delta_G = \sqrt{N} \cdot \delta_L$, where N is the number of steps and δ_L is the local relative error. It is clear that the local relative error δ_L in PsLE method is better criterion to assess global error than the step-size in fixed-step method. The same is true for walk-off method, which in fact, is fixed-step method with automatically adjusted the step-size.

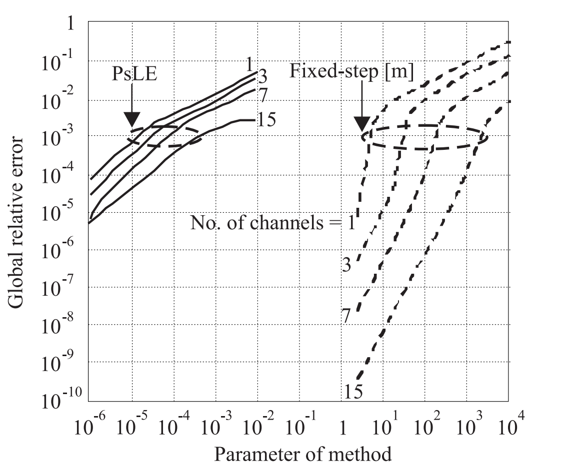

Fig. 3. Global relative error versus parameter of the method: local error for PsLE and step-size for fixed-step.

The PsLE method has two basic advantages: shorter simulation time of up to 50% in comparison with walk-off method, which is known as the most efficient in WDM simulations [5] and offers simply accuracy criterion, i.e., the local error, which is a good indicator of the global accuracy.

3. Role of FWM Spurious Tones on Accuracy of S-SSFM Simulations

Four wave mixing fictitious tones generated during S-SSFM simulations are one of the main sources of errors. Detailed knowledge of their properties is the key factor to improve S-SSFM simulations speed and accuracy.

Actual FWM efficiency η decreases versus the channel separation Δf [1]. Fixed-step S-SSFM with uniform distribution of step-size leads to fictitious FWM efficiency η', presenting several peaks at frequencies f_{p_i}, which was analyzed analytically in [7].

Figure 4 shows the FWM efficiency versus the channel separation Δf after the propagation through a fiber span. The first peak ($\Delta f = f_{p_1}$) on η' curve was shown around 270 GHz. Whatever (signal or noise) is at that spectral distance from a carrier acts like an unrealistic pump for spurious tones. In the walk-off method, uniform step-size distribution is used, in the same way as in the fixed-step method, but the step-size h is adjusted to maintain frequency f_{p_1} of the first fictitious peak at spurious FWM efficiency curve $\eta'(\Delta f)$ outside simulated bandwidth $\Delta f_{\max}$, which is fulfill for $h \ll 1/(2\pi|\beta_2|\Delta f_{\max}^2)$.

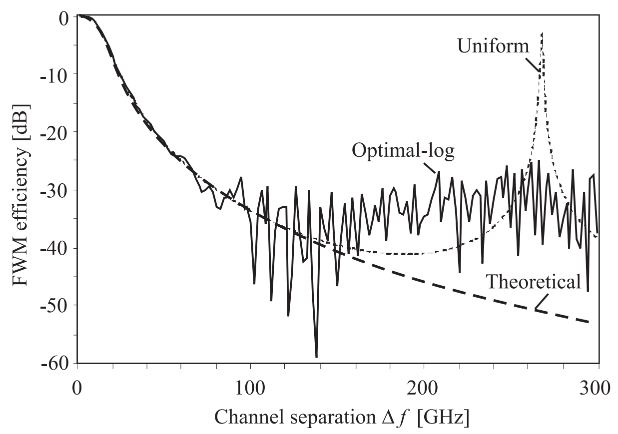

Fig. 4. FWM efficiency as a function of channel separation Δf. True – theoretical, and spurious: for optimal-log and uniform distributions, respectively.

In case of the logarithmic step-size distribution, the FWM spurious distortions η'' follows proper value of η, up to the critical step-size h_{p_1} and then, for higher number of steps K, the η'' behaves like a white noise, with root mean square value inverse-proportional to K. In [7] an analysis was carried out for a simplified case with comb of CW carriers, leading to the following logarithmic step-size distribution:

$$h_n = z_{n+1} - z_n = \frac{1}{2\alpha} \ln\left(\frac{1-nd}{1-(n-1)d}\right), \; n \in \langle 1, K \rangle, \quad (2)$$

where: $d = \frac{1-e^{-2\alpha z}}{K}$, α is the fiber attenuation coefficient and K is the number of steps.

If $\Delta f_{\max} \ll f_{p_1}$, spurious FWM efficiency η' for uniform distribution is only slightly higher than for logarithmic distribution η''. However, step-size h_{p_1} is typically very low (e.g., of the order of 1 m for 15×40 Gbit/s system with 1 nm distance between channels) and larger step-size could be used to obtain global relative error level of 10^{-3}, which is typically sufficient for analysis of WDM system properties [6]. On the other hand, uniform step-size distribution spurious efficiency η' grows sharply for step-size higher than h_{p_1}.

The accuracy gain $\delta_{fix/log}^{\max}$ obtained in S-SSFM simulations with logarithmic step-size distribution compared with uniform one, increases as square root of the number of simulation steps K:

$$\delta_{fix/log} = \sqrt{K} \quad (3)$$

and reaches maximum $\delta^{\max}_{fix/log}$ for the step-size h_{p_1}, corresponding to the resonant frequency f_{p_1}, which is shown in Fig. 5. Additionally, optimal value of parameter A is shown in Fig. 5, which is a slope of logarithmic step-size distribution. The maximum ratio $\delta^{\max}_{fix/log}$ at critical step-size h_{p_1} may exceed 30 dB, which means that the uniform step-size

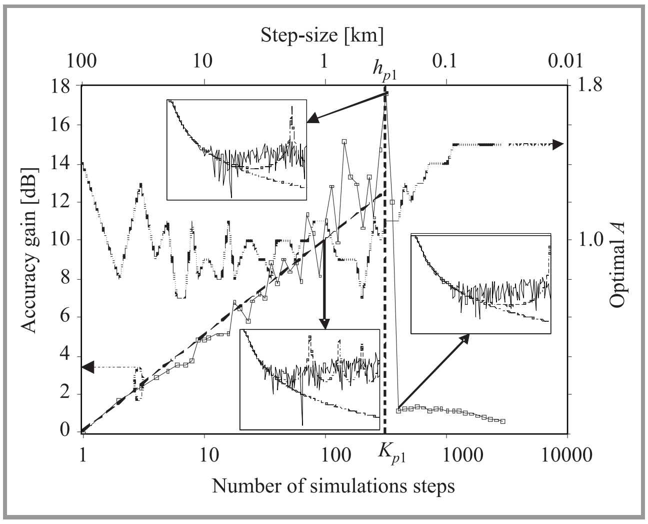

Fig. 5. Accuracy gain of logarithmic distribution over uniform one as a function of number of simulation steps K (or alternatively step-size). Simulation – solid line and theoretical approximation Eq. (3) – dashed line. Insets – FWM efficiencies for a given K.

distribution is not applicable for this step-size, contrary to logarithmic one. Moreover, for step-size far from critical step-size h_{p_1}, e.g., for $5h_{p_1}$, logarithmic distribution is still more accurate than uniform one, for the same number of steps K, and accuracy gain is always consistent with the following limit:

$$\delta^{\max}_{fix/log} \geq \frac{L}{L_{eff}} = \frac{\alpha L}{1 - \mathrm{e}^{-\alpha L}}, \tag{4}$$

where L_{eff} is the fiber effective length.

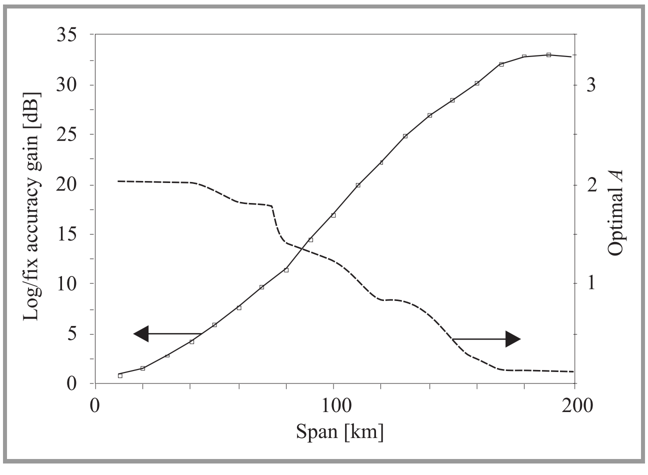

Fig. 6. Accuracy gain of logarithmic distribution over uniform one and optimal value of parameter A as a function of fiber span.

The step-size h is a compromise between the global error δ_G and the simulation time in a real WDM system. In such a system, additional effects, not only FWM, are the source of errors, i.e., SPM and XPM. Moreover, an inter-channel effects (IFWM, IXPM) are generated even in a single channel system.

As can be seen in Fig. 6, optimal value of parameter A, which is a slope of logarithmic step-size distribution, tends to 2 for short simulated fiber spans, and this value has been chosen in [5], which was the source of worsen results of logarithmic distribution, because $A = 2$ is far from optimal value in S-SSFM simulation of actual WDM systems, which is shown in the next section.

4. Modified Logarithmic Step-Size Distribution

Step-size distribution Eq. (2) is used as reference in [5], with conclusion that logarithmic step-size method is somewhat poorer than that of the nonlinear phase and walk-off methods in a single-channel simulations and even further deteriorates in a multi-channel simulations, because the step-size choice is only based on limiting spurious FWM, which is only one of the potential sources of error.

On the other hand, LEM method [5] provides near constant relative local error, which is good strategy to minimize the relative global error, but is slower than the walk-off method (with uniform step-size distribution) due to required parallel calculation of coarse and fine solutions.

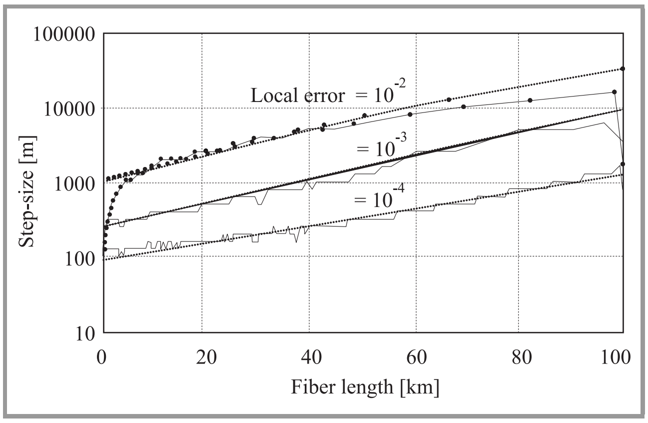

Fig. 7. Step-size distributions obtained in LEM method and its logarithmic approximations for various levels of relative local error (fiber SSMF1, 7 channels, system parameters – see Section 2).

We have found out that the step-size distribution obtained in LEM method is very close to logarithmic, with exception of local fluctuations caused by an algorithm used to maintain the optimal step (see Fig. 7). We have performed several simulations, and each time logarithmic step-size distribution was better than the uniform one, under the assumption that its slope was optimized. Our conclusion is contradiction of that obtained in [5], but in that case not optimal slope of logarithmic step-size distribution was used.

Table 2
Results of S-SSFM simulation for various WDM system scenarios, with the following step-size distributions: uniform, logarithmic obtained by PsLE and optimal logarithmic, for $\delta_G = 2 \cdot 10^{-3}$

Number of channels	Dispersion [ps/(nm·km)]	Span [km]	Number of steps			
			optimal-log (A)	PsLE-log	fixed-step	fixed/log
1	5	100	8 (0.5)	8	16	2.00
3	5	100	122 (0.5)	126	225	1.84
7	5	100	725 (0.6)	740	1400	1.93
15	5	100	3420 (0.5)	3480	6400	1.87
31	5	100	14600 (0.6)	14700	27300	1.87
63	5	100	60000 (0.5)	61000	110600	1.84
1	16	100	17 (0.7)	17	36	2.12
3	16	100	280 (0.7)	300	517	1.85
7	16	100	1600 (0.7)	1780	3150	1.96
15	16	100	7600 (0.7)	7900	14200	1.87
31	16	100	32500 (0.7)	34000	60800	1.87
1	5	50	6 (0.5)	6	8	1.33
3	5	50	92 (0.5)	93	111	1.21
7	5	50	550 (0.5)	560	720	1.31
15	5	50	2570 (0.4)	2590	3150	1.23

It can be shown that when the local signal power is $P(z) = P_0 e^{-\alpha z}$, and the relative local error $\delta(z)$ is proportional to $P(z)^{A\alpha}$, where A is some constant, then the relative local error is uniform in each simulation step, if the following relations:

$$\int_0^{z_1} \delta(z) \mathrm{d}z = \int_{z_1}^{z_2} \delta(z) \mathrm{d}z = \cdots = \int_{z_{K-1}}^{z_K} \delta(z) dz = \frac{1}{K} \int_0^{z_K} \delta(z) dz$$

$$= \frac{1 - e^{-A\alpha z}}{A\alpha K} = \frac{d}{A\alpha}, \quad \text{for} \quad d = \frac{1 - e^{-A\alpha z}}{K}, \qquad (5)$$

are satisfied, which, in turn, occurs when

$$h_n = z_{n+1} - z_n = \frac{1}{A\alpha} \ln\left(\frac{1 - nd}{1 - (n-1)d}\right), \; n \in \langle 1, K \rangle. \quad (6)$$

As can be seen, Eq. (6) is general form of Eq. (2), with additional parameter A, which represents a slope of logarithmic step-size distribution.

Results. The global relative error was calculated for S-SSFM simulation with the following step-size distributions: uniform, logarithmic obtained by PsLE and optimal logarithmic, taking into account various WDM system scenarios. Results are summarized in Table 2.

The optimal value of parameter A for typical simulated WDM systems lays between 0.4 and 0.7 for $\delta_G = 2 \cdot 10^{-3}$, depending of the influence of spurious FWM on the global error. Optimal value of parameter A should be calculated for each simulation and it is time consuming task. PsLE S-SSFM method can be helpful here. In this case, modified logarithmic step-size distribution is a smoothed version of distribution obtained in PsLE S-SSFM pre-simulation. Up to 2 times less steps are needed when optimal logarithmic step-size distribution is used, comparing with walk-off method – known as the most efficient to date.

The optimal logarithmic step-size distribution gives always better results than the uniform one, which is shown in Fig. 8. Logarithmic step-size distribution obtained by pre-simulation local error method is very close to the optimal

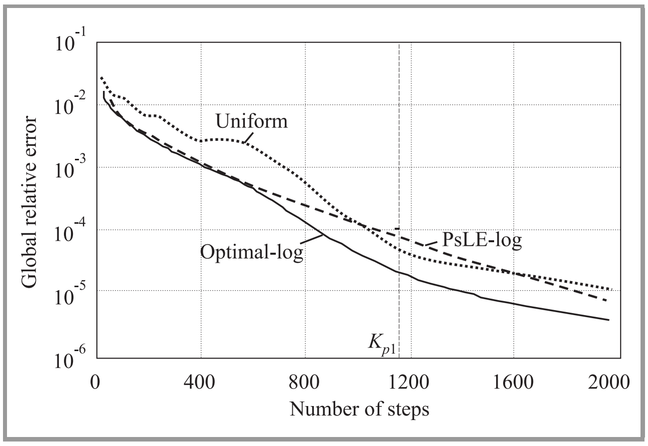

Fig. 8. Global relative error as a function of the number of steps in S-SSFM for various step-size distributions (fiber SSMF1, 3 channels, system parameters – see Section 2).

one in an important global error range of $10^{-2} - 10^{-3}$, but for lower levels of global error the results is even slightly worse than for the uniform distributions, due to the bigger than optimal value of the parameter A, which occurs for global relative error lower than $5 \cdot 10^{-4}$ (see Fig. 9).

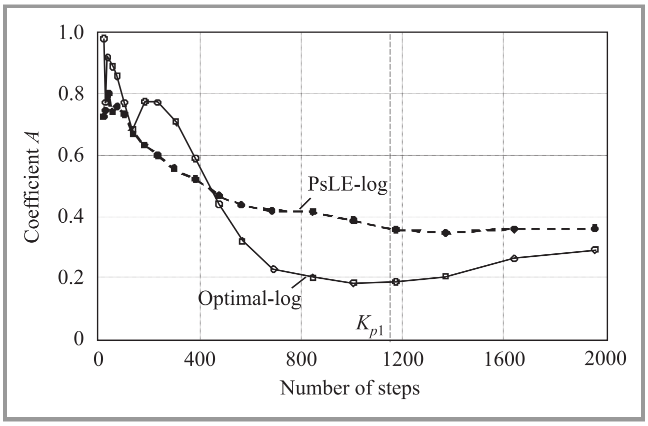

Fig. 9. Optimal and obtained by PsLE method, coefficient A of logarithmic step-size distributions as a function of the number of steps in S-SSFM (fiber SSMF1, 3 channels, system parameters – see Section 2).

Dependence between the relative global error and the coefficient A is presented in Fig. 10.

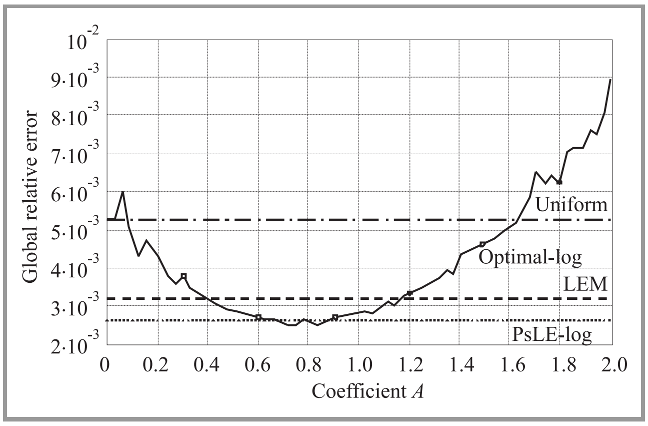

Fig. 10. Relative global error as a function of coefficient A for S-SSFM simulation. Results of fixed-step (uniform) and LEM methods are presented for comparison (fiber SSMF1, 7 channels, system parameters – see Section 2).

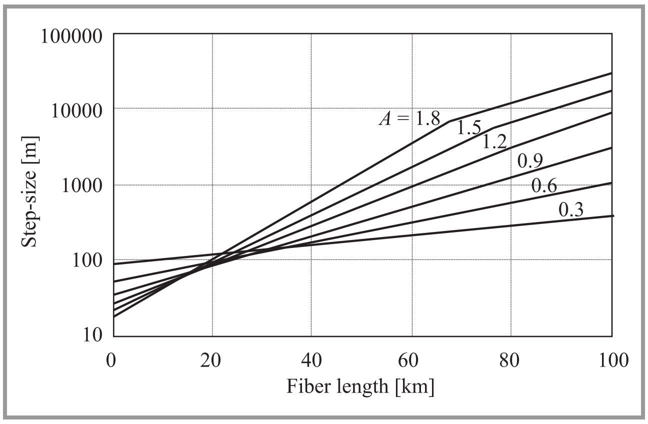

Fig. 11. Step-size distributions as a function of fiber length for various values of parameter A (100 km of fiber with $\alpha = 0.22$).

As can be seen in Fig. 10 optimal value of parameter A lays between 0.5–1.0 and for $A = 2$ used in [5], error is two times higher than obtained for uniform step-size distribution. The step-size distributions corresponding to various values of parameter A, for 100 km of fiber with $\alpha = 0.22$, are shown in Fig. 11.

As a rule of thumb, logarithmic step-size distribution improves global relative accuracy by

$$\Delta\delta_G = \frac{L}{L_{eff}} = \frac{\alpha L}{1 - e^{-\alpha L}}$$

as compared to uniform step-size distribution, which is illustrated in Fig. 12.

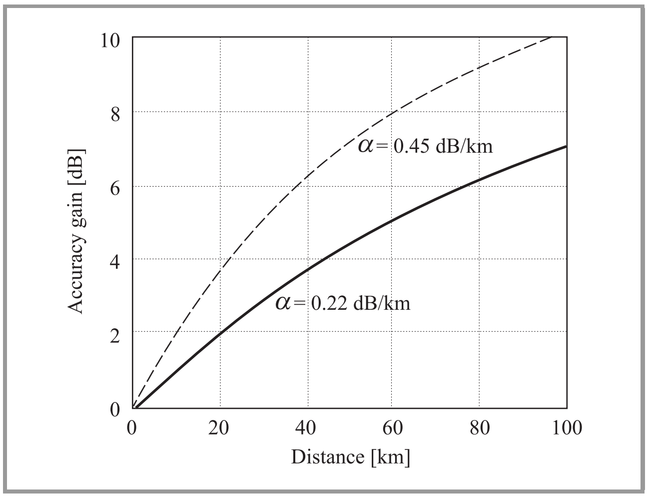

Fig. 12. Accuracy gain as a function of the distance for uniform step-size distribution.

Our future work is concentrated on finding more accurate and faster methods to chose optimal values of the parameter A and the number of steps K, needed for a given relative global error δ_G.

5. Conclusions

Pre-simulated local error S-SSFM typically halves simulation time of WDM links, comparing to conventional fixed-step S-SSFM. Moreover, local error used in pre-simulation seems to be a good indicator of the global accuracy. Even more effective step-size distribution can be achieved using modified logarithmic method, although in this case, methods to found the optimal value of slope for logarithmic step-size distribution and the number of steps, for a given global accuracy, should be further studied. To the best of our knowledge proposed two novel methods are faster than other methods for simulations of light propagation in WDM links. Up to 2 times less steps are needed when optimal logarithmic step-size distribution is used, comparing with walk-off method – known as the most efficient until now.

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Marek Jaworski was born in Warsaw, Poland, in 1958. He received the M.Sc. degree in electronic engineering from the Warsaw University of Technology in 1982 and a Ph.D. degree in communications engineering from the National Institute of Telecommunications (NIT) in Warsaw, in 2001. He has been with NIT since 1982, working on modeling and design of optical fiber transmission systems, measurement methods and test equipment for optical networks. He has been engaged in several European research projects since 2003, including COST Actions 270, 291 and 293. This included research of polarization mode dispersion in optical fiber, numerical simulations of telecommunication systems, advanced modulation formats and nonlinear photonics. Dr. Jaworski is an author of 3 Polish patents and 25 scientific papers in the field of optical fiber communications and measurements in optical networks, as well as one of the "Journal of Telecommunications and Information Technology" associate editors.

e-mail: M.Jaworski@itl.waw.pl
National Institute of Telecommunications
Szachowa st 1
04-894 Warsaw, Poland